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Solutions of hyperbolic telegraph equation using radial basis function-finite difference method

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Abstract This study introduces a novel Radial Basis Function–Finite Difference (RBF-FD) numerical framework for the solution of the hyperbolic telegraph equation, a fundamental model arising in reaction–diffusion phenomena and wave–diffusion processes. The proposed scheme employs a fully meshless formulation based on polyharmonic spline radial basis functions augmented with polynomials, thereby eliminating the need for shape-parameter selection that commonly hampers classical RBF methods. This design ensures enhanced numerical stability, and high-order accuracy while retaining the inherent flexibility of meshless discretization for handling irregular and complex geometries. Unlike traditional grid-based approaches, the presented RBF-FD method seamlessly accommodates scattered node distributions and time-dependent dynamics. The effectiveness and robustness of the scheme are demonstrated through four benchmark test problems, where numerical solutions exhibit excellent agreement with available analytical solutions. Detailed error analyses confirm consistently negligible errors across all examples. These results establish the RBF-FD approach as an efficient and versatile computational tool for solving hyperbolic-type partial differential equations on complex domains.

Keywords: Hyperbolic telegraph equation, Numerical Solution, Radial basis function, Finite Difference method

1 Introduction

The Radial Basis Function–Finite Difference (RBF-FD) method is an innovative numerical technique for solving differential equations. It was first introduced by Rolland Hardy in 1971. This approach has been extensively used since its beginning to tackle a variety of problems in several disciplines, including applied sciences, engineering, and geophysics (Martin and Fornberg 2017, Li 2017, Ćecile 2019). Readers and scholars interested in exploring the fundamental concepts and various applications of the RBF-FD technique can find a wealth of comprehensive references in the literature (Fornberg and Flyer 2015, Bayona 2019, Mudiyanselage 2022).

The RBF approach was first used mainly as an interpolation method to address application-focused issues (Du Toit 2008, Estruch *et al*. 2013). This approach offers the advantage of not requiring the integral evaluation as a meshless method based on a collocation scheme. However, one of the drawbacks of the classical RBF method is the selection of an optimal shape parameter, which significantly influences accuracy and stability (Mongillo 2011). To address this issue, some researchers have adopted thin plate spline RBFs. Nevertheless, this option has its own disadvantage since it does not ensure that the collocation matrices produced by the RBF method are non-singular.

The use of RBF-based techniques to address challenging differential equations based on scientific and engineering problems has increased in popularity throughout the last ten years. A researcher called Kansa further advanced this methodology by developing a collocation-based RBF approach for approximating solutions of partial differential equations (PDEs) (Usta 2016). Even with its effectiveness, the RBF method suffers from severe ill-conditioning of the collocation matrix when applied to large-scale problems. To mitigate this, the RBF-MQ method was introduced (Ouédraogo *et al*. 2019). However, this technique still has issues with parameter sensitivity when employing with infinitely smooth RBFs.

The RBF-FD approach significantly alleviates ill-conditioning and parameter dependency issues, making it a powerful tool for solving large-scale PDEs efficiently and accurately by combining the flexibility of RBF interpolation with the sparsity and stability properties of finite difference schemes. The present work demonstrates that the RBF-FD numerical scheme can be applied effectively to study hyperbolic telegraph equations. The test problems considered in this study were adopted from the literature reported by Dehghan and Shokri (2007), where the equation was originally solved using Kansa's global collocation method. These benchmark problems were used to evaluate the accuracy and performance of the RBF–FD approach.

2 Material and Methods

The following hyperbolic telegraph equation was adopted from literature (Dehghan and Shokri 2007, Hashemi *et al*. 2019).

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u = \frac{\partial^2 u}{\partial x^2} + h(x,t), \quad x \in \ \Omega = [a,b] \subset \mathbb{R}, \quad 0 < t < T, \tag{1}$$

with initial condition

$$\begin{cases} u(x,0) = f_1(x) & x \in \Omega \\ u_t(x,0) = f_2(x) & x \in \Omega, \end{cases} \tag{2}$$

and Dirichlet boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \qquad 0 < t \leq T \tag{3}$$

Here h, f_1, f_2 and g are known functions while α and β are constant coefficients. The objective of this study is to determine the function $u(x,t)$ using the RBF-FD method.

The following section explains the application of the RBF-FD method. The proposed methodology was validated through four test examples.

Applying the Backward Euler formula, we obtained the following expressions for the first, and second derivatives of $u(x,t)$ with respect to time.

$$u(x, t+\Delta t) \approx \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} \quad (4)$$

$$\frac{\partial^2}{\partial t^2} u(x, t+\Delta t) \approx \frac{u(x, t+\Delta t) - 2u(x,t) + u(x, t-\Delta t)}{(\Delta t)^2} \quad (5)$$

Then the equation (1) was written as follows.

$$\frac{u(x, t+\Delta t) - 2u(x,t) + u(x, t-\Delta t)}{(\Delta t)^2} + \alpha\left(\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}\right) + \beta u(x, t+\Delta t) = D_{lap} u(x, t+\Delta t) + h(x, t+\Delta t) \quad (6)$$

The equation (6) was rearranged using the notation $u(x, t^n) = u^n$, where $t^n = t^{n-1} + \Delta t$.

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} + \alpha\left(\frac{u^{n+1} - u^n}{\Delta t}\right) + \beta u^{n+1} = D_{lap} u^{n+1} + h(x, t^{n+1}) \quad (7)$$

By applying the finite difference scheme for the time derivatives of $u(x,t)$, approximations for $\frac{\partial^2 u}{\partial t^2}$, and $\frac{\partial u}{\partial t}$ were obtained, and equation (7) was derived. In contrast, Niknam and Adibi (2022) employed a Crank–Nicolson time integration scheme for all derivatives of $u(x,t)$ in the equation (1). In the present approach, however, the finite difference scheme was applied exclusively to the time derivatives in the equation (1). The operator $D_{lap} = \frac{\partial^2}{\partial x^2}$ denotes the Laplacian, which is approximated using the RBF–FD framework. For this purpose, the function $u(x,t)$ was expressed as a linear combination of radial basis functions and polynomial terms as in equation (8).

$$u(x,t) = \sum_{i=1}^{n} \lambda_i \, \phi(||x - x_i||) + \sum_{i=1}^{M} \mu_i \, p_i(x_i) \quad (8)$$

where n is the number of neighbors around the center node x_c, and λ_i and μ_i are unknown values. This system could be represented in matrix form as follows.

$$\left(\tilde{C} = \begin{bmatrix} C & P \\ P^T & O \end{bmatrix}\right) \begin{bmatrix} \lambda \\ \underline{\mu} \end{bmatrix} = \begin{bmatrix} u \\ \underline{0} \end{bmatrix} \quad (9)$$

where C is the collocation matrix defined as,

$$C = \begin{pmatrix} \phi(||x_1 - x_1||) & \phi(||x_1 - x_2||) & \dots & \phi(||x_1 - x_n||) \\ \phi(||x_2 - x_1||) & \phi(||x_2 - x_2||) & \dots & \phi(||x_2 - x_n||) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(||x_n - x_1||) & \phi(||x_n - x_2||) & \dots & \phi(||x_n - x_n||) \end{pmatrix}$$

and P is the transpose of Vandermonde matrix given as follows.

$$P = \begin{pmatrix} P_1(x_1) & P_2(x_1) & P_3(x_1) & P_\sigma(x_1) \\ P_1(x_2) & P_2(x_2) & P_3(x_2) & P_\sigma(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_1(x_n) & P_2(x_n) & P_3(x_n) & P_\sigma(x_n) \end{pmatrix}$$

Next, the operator $D_{lap} = \frac{\partial^2}{\partial x^2}$ is applied to the equation (8) for finding the $D_{lap}u^{n+1}$ in the equation (7). This led to equation (10).

$$D_{lap}u(x,t) = \sum_{i=1}^{n} \lambda_i \, D_{lap}\phi(||x - x_i||) + \sum_{i=1}^{M} \mu_i \, D_{lap}p_i(x_i) \quad (10)$$

We note that, λ_i and μ_i were not changed when applying operator to the equation (8). The λ_i and μ_i could be found using the equation (9) while matrix $\tilde{C}$ is nonsingular. Substituting these values into the equation (10) yielded the equation (11):

$$D_{lap}u(x,t) = \left(\begin{bmatrix} \nabla\phi_{x=x_c} & \nabla p_{x=x_c} \end{bmatrix}_{1\times(n+M)} \begin{bmatrix} C & P \\ P^T & O \end{bmatrix}^{-1}_{(n+M)\times(n+M)} \right) \begin{bmatrix} u \\ \underline{0} \end{bmatrix} \quad (11)$$

The right-hand side of the equation (11) is referred to as the differentiation matrix. By applying appropriate boundary conditions and initial condition, the resulting linear system (11) and (7) were solved using the MATLAB software. In this study, MATLAB was used to implement the methodology as a code capable of generating graphical solutions for all test examples. It is noted that the computational complexity of this process was $O(n^2N)$. Furthermore, the numerical solution was achieved to an accuracy of $O(h^{p-1})$ when using the RBF-FD approach with an augmentation polynomial of order p according to Fornberg and Flyer (2015) and Bayona (2019).

3 Results and Discussion

RBF–FD method within a meshless finite-difference framework is proposed to approximate the hyperbolic telegraph equations. Time derivatives were evaluated using a finite-difference formulation, while spatial operators were discretized through the RBF–FD approach. The resulting algebraic system closely resembles classical finite-difference schemes, yet eliminates the requirement for structured grids. To validate the proposed method, four benchmark examples from the literature (Dehghan and Shokri 2007) were considered to validate the proposed method. For each example, numerical results were compared with the corresponding exact analytical solutions, and errors were quantified using the root mean square error (RMSE).

3.1 Example 1

The hyperbolic telegraph equation (1) was solved by considering $\alpha = 1,\ \beta = 1$ and $h(x,t) = 0$ in the interval $0 \leq x \leq 4$. The initial condition was given below:

$$u(x,0) = e^x, \qquad u_t(x,0) = -e^x, \quad 0 \leq x \leq 4 \tag{12}$$

For validating the obtained numerical results, the exact solution given in equation (13) was considered from Momani 2005, Dehghan and Shokri 2007).

$$u(x,t) = e^{x-t} \tag{13}$$

For this example, equation (7) can be simplified by taking $h(x,t) = 0$, resulting in

$$\left(\frac{1}{\Delta t^2} + \frac{1}{\Delta t} + 1 - D_{lap}\right) u^{n+1} = \left(\frac{2}{\Delta t^2} + \frac{1}{\Delta t}\right) u^n - \frac{1}{\Delta t^2} u^{n-1} \tag{14}$$

To find u^{n+1}, the Dirichlet boundary conditions are imposed by using the exact solution.

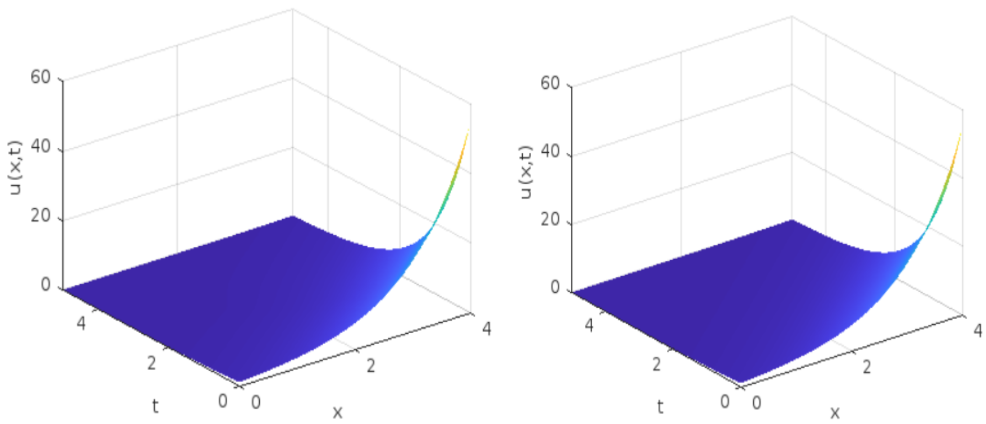

Fig 1. Numerical (left) and analytical (right) solutions obtained for the Example 1 using 200 spatial nodes

Figure 1 shows the numerical solution (left) obtained using the RBF-FD method, and the analytical solution (right) taken from the cited paper (Dehghan and Shokri 2007) for the Example 1. Both numerical and analytical solutions were illustrated up to $t = 5s$, using 200 spatial nodes. No significant deviation is visible in the Figure 1 when comparing the analytical and numerical results. RMSE values related to the numerical solutions over time in the domain [0, 4] for example are presented in Table 1, and they confirm the high accuracy and reliability of the proposed solution due to their small magnitudes.

Table 1. RMSE values of numerical solutions over time in the domain [0, 4] for the Example 1

t (s)	RMSE
1	6.3408 x 10^{-3}
2	4.8941 x 10^{-3}
3	1.5915 x 10^{-3}
4	7.3787 x 10^{-4}
5	8.2153 x 10^{-4}

Furthermore, the accuracy of the obtained solution was investigated by varying the number of nodes (N), as represented in Figure 2. There were no significant differences of the solution when changing the number of nodes from 50 to 300 (lines are coincided). These results consistently indicate that the numerical solutions produced by the RBF-FD method remain in excellent agreement with the analytical solution, confirming the robustness and effectiveness of the method.

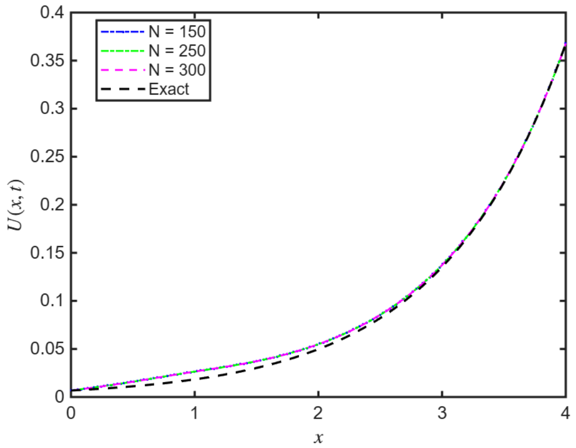

Fig 2. The comparison of numerical solutions of $u(x,t)$ with the exact solution in $t = 5s$ for the example 1 with different number of nodes (N)

3.2 Example 2

The hyperbolic telegraph equation (1) on the interval $0 \leq x \leq \pi$ has the analytical solution (Gao and Chi 2007): $u(x,t) = e^{-t}\sin x$, that is a combination of exponential and trigonometric functions. The initial conditions were as follows.

$$u(x,0) = \sin x, \quad u_t(x,0) = -\sin x, \quad 0 \leq x \leq \pi \qquad (12)$$

The equation (13) was obtained by simplifying equation (7) with $\alpha = 6$, $\beta = 2$ and $h(x,t) = (2 - \alpha + \beta)e^{-t}\sin x$, related to the Example 2.

$$\left(\frac{1}{\Delta t^2}+\frac{6}{\Delta t}+2-D_{lap}\right)u^{n+1}=\left(\frac{2}{\Delta t^2}+\frac{6}{\Delta t}\right)u^{n}-\frac{1}{\Delta t^2}u^{n-1}-2e^{-t^{n+1}}\sin x \qquad (13)$$

The numerical (left) and analytical (right) solutions are graphically shown in Figure 3, which were obtained by solving the equation (13) using MATLAB software. The numerical results were validated through comparison with the analytical solution, demonstrating the accuracy and effectiveness of the RBF-FD method.

Figure 3 indicates that the RBF–FD approach can be effectively applied to solve hyperbolic telegraph equations involving combinations of trigonometric and exponential functions.

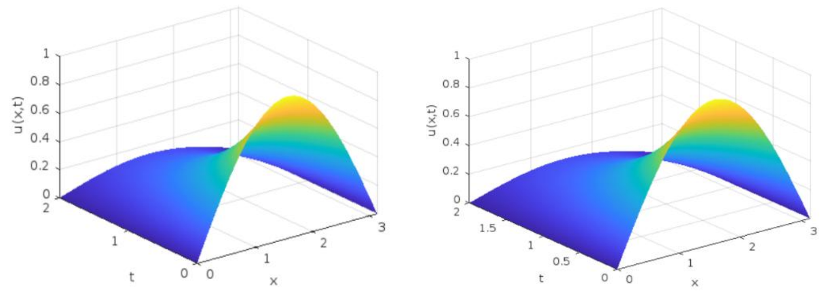

Fig 3. Numerical (left) and analytical (right) solutions obtained for the Example 2 using 200 spatial nodes

The numerical and analytical solutions at t = 5 are illustrated in Figure 4 for different numbers of nodes. The variation between the numerical and analytical solutions could be observed more clearly. There were no significant differences of the numerical solution when changing the number of nodes from 150 to 300. Since numerical methods provided approximate solutions, they inherently introduce some error with respect to the analytical solution.

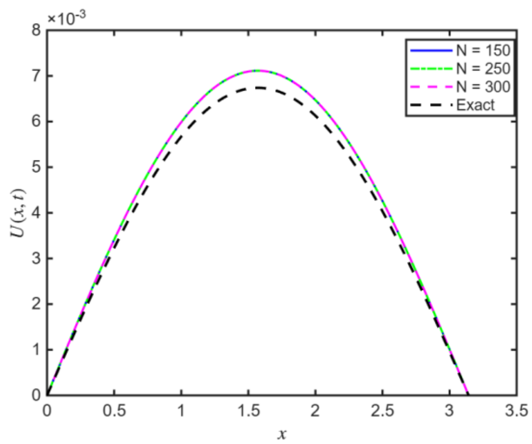

Fig 4. The comparison of numerical solutions of $u(x,t)$ with the exact solution in $t = 5s$ for the Example 2 with different number of nodes (N)

Numerical solutions exhibit only a negligible error when compared to the analytical solution, even when the solution contains both trigonometric and exponential components (Figure 4 and Table 2). The RMSE values of the numerical solutions further validate the accuracy and reliability of the results obtained using the proposed RBF–FD approach.

Table 2. RMSE values of numerical solution over time in domain $[0, \pi]$ for Example 2

t (s)	RMSE
1	1.0074 x 10^{-5}
2	6.0964 x 10^{-5}
3	8.8151 x 10^{-5}
4	1.6455 x 10^{-4}
5	8.2153 x 10^{-4}

3.3 Example 3

The hyperbolic telegraph equation (1) has the exact solution (El-Azab and El-Gamel 2007) in the interval $0 \leq x \leq 1$, with constant coefficients $\alpha = 1, \beta = 1$ and $h(x,t) = (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2e^{-t}$ as

$$u(x,t) = (x - x^2)t^2e^{-t} \tag{14}$$

In this example, the RBF-FD approach was employed to approximate the second derivative operator with respect to x. Furthermore, the resulting system given in equation (15) was obtained by simplifying the equation (7) with Backward Euler formula.

$$\left(\frac{1}{\Delta t^2} + \frac{1}{\Delta t} + 1 - D_{lap}\right)u^{n+1} = \left(\frac{2}{\Delta t^2} + \frac{1}{\Delta t}\right)u^n - \frac{1}{\Delta t^2}u^{n-1} + (2 - 2t^{n+1} + t^{2(n+1)})(x - x^2)e^{-t^{n+1}} + 2t^{2(n+1)}\,e^{-t^{n+1}} \tag{15}$$

For the Example 3, following results were obtained considering the time step $\Delta t = 0.025$ and applying Dirichlet boundary conditions generated from the actual solution.

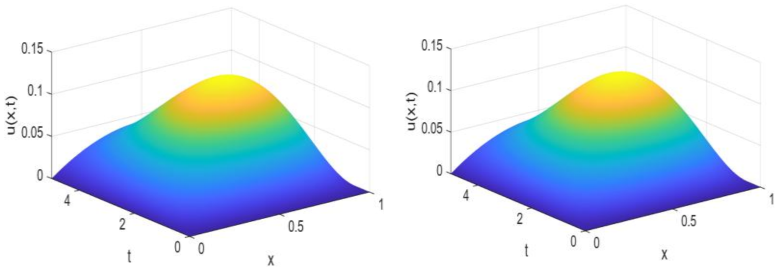

Fig 5. Numerical (left) and analytical (right) solutions obtained for Example 3 are presented for 200 nodes

The left and right figures in Figure 5 illustrate the numerical and analytical solutions of $u(x,t)$ in $t = 5s$ and $N = 200$ for the Example 3. The numerical results in Figure 6 are obtained using different numbers of spatial nodes $N = 150$ and 300 represented by distinct line colours. The exact solution is depicted with a black dashed line. It can be observed that all numerical solutions closely match the exact solution.

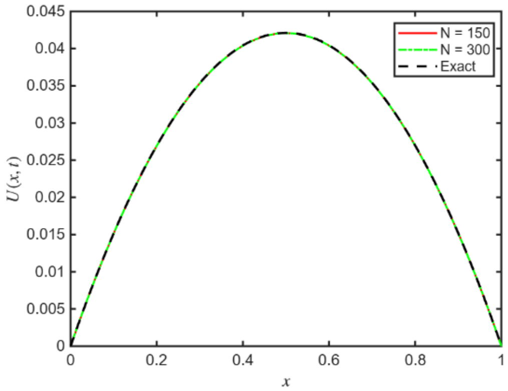

Fig 6. The numerical solution and exact solution of $u(x,t)$ in $t = 5s$ for the Example 3

However, from Figure 6, it is difficult to identify any significant changes in the numerical solution as the number of nodes (N) increases. This demonstrates the convergence of the proposed numerical method toward the exact solution. Moreover, the small error values again validated the numerical results presented in Table 3.

Table 3: RMSE values of numerical solution over time in domain [0, 1] for example 3.

t (s)	RMSE
1	6.4446×10^{-5}
2	4.7250×10^{-5}
3	4.2209×10^{-5}
4	2.6855×10^{-5}
5	4.3738×10^{-6}

3.4 Example 4

As the last example, the hyperbolic telegraph equation (1) was considered with $\alpha = 1$, $\beta = 1$ and $h(x,t) = x^2 + t - 1$ in the interval $0 \leq x \leq 1$. The initial conditions were given by

$$u(x,0) = x^2, \qquad u_t(x,0) = 1, \qquad 0 \leq x \leq 1, \tag{16}$$

with the analytical solution:

$$u(x,t) = x^2 + t \tag{17}$$

By modifying the MATLAB code for RBF-FD according to this example, obtained the following results.

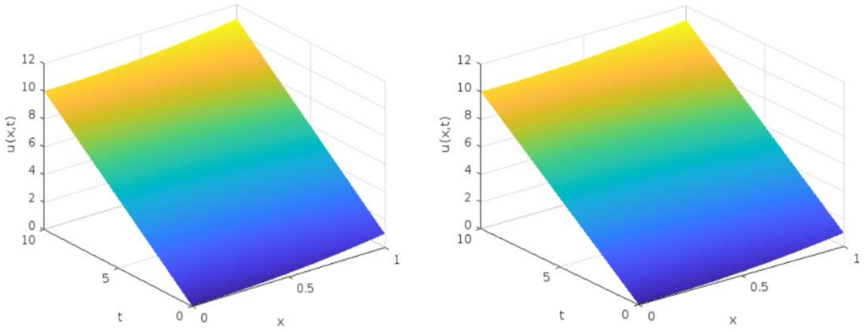

Fig 7. The numerical (left) and analytial (right) results of $u(x,t)$ are represented for the Example 4 with 200 nodes.

Figure 7 presents the comparison between the analytical (left) and numerical (right) results of $u(x,t)$ obtained using the RBF-FD method. The Figure 8 shows a cross-sectional comparison at a specific time t. The numerical results were computed for different number of nodes N, and compared with the analytical solutions. These findings clearly show that the numerical solutions closely follows the analytical solutions for all tested values of N. The RBF-FD approach indicates adequate precision in approximating the solution, as evidenced by the minimal deviations between the numerical and analytical solutions.

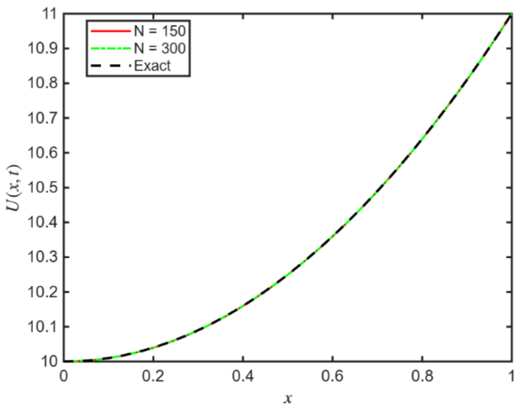

Fig 8. Numerical and analytical results of $u(x,t)$ in $t = 10s$ for the example 4 with different values of N.

Furthermore, RMSE error values in Table 4 verify the accuracy of the numerical approach. The accuracy of numerical solution with respect to the analytical solution confirms the strength of the RBF-FD method for solving these types of problems.

Table 4: RMSE values of numerical solution over time in domain [0, 1] for example 4.

t (s)	RMSE
1	3.8305×10^{-4}
2	1.0172×10^{-3}
3	1.7417×10^{-3}
4	2.9532×10^{-5}
5	4.3738×10^{-6}

To assess the numerical stability of the proposed method, additional experiments were conducted by varying the time-step size while keeping the spatial discretization fixed, with all computations performed in MATLAB. The results indicate that no significant change in the solutions across different time steps. Furthermore, Figures 2, 4, 6, and 8 illustrate that the numerical solutions remain stable for varying node distributions in the benchmark problems. In all cases, the solutions are bounded and free from spurious oscillations, even over long-time integration, indicating that the proposed method exhibits stable numerical behavior.

The main findings of this study demonstrate that the RBF–FD method provides accurate numerical solutions for the hyperbolic telegraph equation while significantly improving computational efficiency compared with global RBF methods. Because the method relies on local stencils rather than global interpolation, the resulting differentiation matrices are sparse, thereby reducing both computational cost and memory requirements. In addition, the method shows stable performance even when nodes are irregularly distributed, highlighting its flexibility for practical numerical simulations. Polyharmonic RBFs are employed to generate the numerical solutions of the hyperbolic telegraph equation, which helps overcome the shape-parameter selection issue commonly encountered in traditional RBF methods and ensures the non-singularity of the collocation matrix.

Niknam and Adibi (2022) also investigated the solution of the hyperbolic telegraph equation using a local radial basis function (LRBF) collocation method combined with a Crank–Nicolson time integration scheme, where the spatial discretization was carried out through local RBF interpolation. In their study, the shape parameter was assumed to be constant, however, selecting an optimal shape parameter remains a well-known challenge in RBF-based methods and can significantly affect solution accuracy and stability. Although some researchers (Dehghan and Shokri 2007) addressed this issue by employing thin-plate spline RBFs, which eliminate the need for a shape parameter, their approach still results in dense collocation matrices and may become inefficient for problems that are more complex.

4 Conclusions

This study presented a novel numerical scheme based on the RBF-FD approach to solve one-dimensional hyperbolic telegraph equation. This numerical scheme incorporates the Backward Euler formula to find the first and second derivatives of $u(x,t)$ with time. By addressing the parameter-related limitations of previous methods, this study presented an approach to approximate the Laplacian operator, D_{lap} in the hyperbolic telegraph equation using the RBF–FD method. The proposed scheme was applied successfully to four benchmark test problems to verify its validity. Numerical solutions obtained using MATLAB were validated by comparison with the corresponding analytical solutions, demonstrating the accuracy and effectiveness of the RBF–FD approach. Furthermore, the error tables highlighted the accuracy of the RBF-FD approach, demonstrating its effectiveness in solving hyperbolic telegraph nonlinear partial differential equations.

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